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Directed Random Walks in Random Environments

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We use holding time methods to study the asymptotic behavior of pure birth processes with random transition rates. Both the normal and slow approaches to infinity are studied. Fluctuations are shown to obey the central limit theorem for almost all sample-transition rates. Our results are stronger, and our proofs simpler, then those of recently published studies.

KEY WORDS: Directed random walk; pure birth process; holding time; strong law of large numbers; central limit theorem.

1. INTRODUCTION

Random walks in random environments have received much attention from mathematical physicists in recent years,(1-10) mainly as models for the motion of electrons in crystals with impurities. The presence of the defects perturbs the normal hopping behavior of the electrons from one ion of the crystal to the next, thus modifying the transport properties of the medium. Because the nature and location of the defects can only be controlled in a statistical sense, their effect is best taken into account by treating the *transition rates* of the walk as *random variables.*

The main objects of interest relate to the asymptotic properties of the stochastic process X_t describing the position of the particle at time t. For instance, we would like as precise a description as possible of

$$\lim_{t \to \infty} X_t/t \tag{1}$$

and

$$\lim_{t \to \infty} t^{-1/2}(X_t - \mathbf{E}[X_t]) \tag{2}$$

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Expression (1) can be interpreted as the asymptotic velocity of the particle and is related to the electric conductivity of the disordered medium. Expression (2) describes fluctuations. It is of special importance to identify those quantities that are *sample independent*, that is, independent of the particular realization of the disorder.

In a series of recent articles, Aslangul *et al.*[4-6] have studied the very special case where the random walk is one-dimensional and constrained to jump to the right only (in the accepted terminology, this is a *pure birth process*). Accordingly, they consider the following set of differential equations for the probability $P_n(t)$ that at time t the process X_t is in state $n \in \mathbf{N}$:

$$\begin{aligned} P'_n(t) &= -W_n P_n(t) + W_{n-1} P_{n-1}(t), \qquad n \geqslant 1 \\ P'_0(t) &= -W_0 P_0(t) \end{aligned} \tag{3}$$

where $\underline{W} = \{W_j\}_{j=0}^{\infty}$ is a sequence of independent identically distributed nonnegative random variables. Using generating functions and rather opaque asymptotic expansions, they obtain the following results:

For almost every sample $\underline{w} = \{w_j\}_{j=0}^{\infty}$ taken from $\underline{W} = \{W_j\}_{j=0}^{\infty}$ the following limits hold:

$$\lim_{t \to \infty} \frac{1}{t} \mathbf{E}_{\underline{w}}[X_t] = (\mathbf{E}[W_0^{-1}])^{-1} \tag{4}$$

$$\lim_{t \to \infty} \frac{1}{t} \mathbf{D}_{\underline{w}}[X_t] = \mathbf{E}[W_0^{-2}](\mathbf{E}[W_0^{-1}])^{-3} \tag{5}$$

In these formulas, as everywhere in this article, we index by $\underline{w}$ all the probabilistic quantities computed for a fixed sample $\underline{w}$ of $\underline{W}$. Averaging over the W_j is denoted by $\mathbf{E}[\cdot]$. Formulas (4) and (5) show that the *mean* asymptotic velocity and the asymptotic variance of $X_t/\sqrt{t}$ are sample-independent. The method of proof used in ref. 4 is so involved that a full proof of (5) is not included and that even the proof of the simpler formula

$$\lim_{t \to \infty} \frac{1}{t} \mathbf{E}\{\mathbf{D}_{\underline{W}}[X_t]\} = \mathbf{E}[W_0^{-2}](\mathbf{E}[W_0^{-1}])^{-3} \tag{6}$$

is said (again without details) to depend on the properties of modified Bessel functions.

In this article we use *holding times* to obtain straightforward proofs of formulas that are *stronger* than (4) and (5), namely:

For almost samples $\underline{w}$

$$\lim_{t \to \infty} X_t/t = (\mathbf{E}[W_0^{-1}])^{-1} \qquad \text{for almost all sample paths of } X_t \tag{7}$$

$$\lim_{t \to \infty} \mathbf{P}_{\underline{w}} \left[\frac{X_t - \mathbf{E}_{\underline{w}}[X_t]}{\sqrt{t}} \leqslant x \right] = \Phi_{\sigma^2}(x) \tag{8}$$

where $\Phi_{\sigma^2}(x)$ is the normal probability distribution function with zero mean and variance σ^2

$$\Phi_{\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2\sigma^2}\, dy \tag{9}$$

$$\sigma^2 = \mathbf{E}[W_0^{-2}](\mathbf{E}[W_0^{-1}])^{-3} \tag{10}$$

Formula (7) shows that the averaging over all paths in (4) is unnecessary, whereas (8) gives the full limiting distribution of the fluctuations.

Finally, it is of interest to study what happens when $\mathbf{E}[W_0^{-1}] = \infty$. In that case the rate of divergence of X_t to infinity is always slower than linear. The characterization of this rate is delicate (see Section 4), but can be described roughly as follows:

If for some $0 < \mu < 1$, $\mathbf{E}[W_0^{-\mu}] = \infty$ but $\mathbf{E}[W_0^{-\mu+\varepsilon}] < \infty$ for all $\varepsilon > 0$, then t^μ is the rate of approach of X_t to infinity, in the sense that the limit of X_t/t^α is zero if $\alpha > \mu$ and infinity if $\alpha < \mu$. The description of the limiting behavior of X_t/t^μ requires a more detailed knowledge of the distribution of W_0^{-1} (see Section 4).

To conclude this introduction, we direct the reader to a previous study of a related problem: a *discrete-time birth and death* process with random transition rates.[11,12] Superficially, the asymptotic results obtained in these articles resemble ours, but the authors do not (as we do) establish limit laws that hold *for almost all environments.* A pertinent physical discussion of the various concepts of limit laws for random walks in random environments can be found in ref. 13.

2. PRELIMINARY: THE PURE BIRTH PROCESS

In this section, we gather well-known results about the pure birth process.[14–16] This is the continuous-time integer-valued Markov process $\{X_t, t \geqslant 0\}$ defined by the transition probabilities

$$\mathbf{P}_{\underline{w}}[X_t = n \mid X_0 = 0] = P_n(t), \qquad \mathbf{P}[X_0 = 0] = 1 \tag{11}$$

where

$$\begin{aligned} P_n'(t) &= -w_n P_n(t) + w_{n-1} P_{n-1}(t), \qquad n \geqslant 1 \\ P_0'(t) &= -w_0 P_0(t) \end{aligned} \tag{12}$$

In (12) the w_j, $j = 0, 1, 2,...,$ are given positive numbers. In words, the only allowed transition out of state n is to state $n+1$, and this occurs with rate w_n.

An equivalent description of this process can be given in terms of *holding times*(14–16): once the process has reached state n, it stays there for a random time T_n, after which it jumps to $n+1$. The holding times T_0, T_1, T_2,... are independent random variables which are exponentially distributed with parameters w_0, w_1, w_2,...:

$$\mathbf{P}_{\underline{w}}[T_j > x_j, 0 \leqslant j \leqslant n] = \exp\left(-\sum_{j=0}^{n} w_j x_j\right) \tag{13}$$

The connection between the two viewpoints is made by the formula

$$X_t = \max\{n\colon S_n \leqslant t\} \tag{14}$$

where $S_0 = 0$ and

$$S_n = \sum_{j=0}^{n-1} T_j, \qquad n \geqslant 1 \tag{15}$$

is the time of the nth jump. A sample path $\{x_t, t \geqslant 0\}$ of the pure birth process and the corresponding realization $\{t_j\}_{j=0}^{\infty}$ of the sequence of holding times are represented in Fig. 1.

Note that, from (14), a sample path x_t reaches infinity in a finite time τ if and only if infinitely many jumps occur during $[0, \tau)$, i.e., $\sum_{j=0}^{\infty} t_j = \tau < \infty$.

Using the fact that the random variables T_j are independent and exponentially distributed, it is easy to check that the set of such paths has probability zero if and only if

$$\sum_{j=0}^{\infty} w_j^{-1} = \infty \tag{16}$$

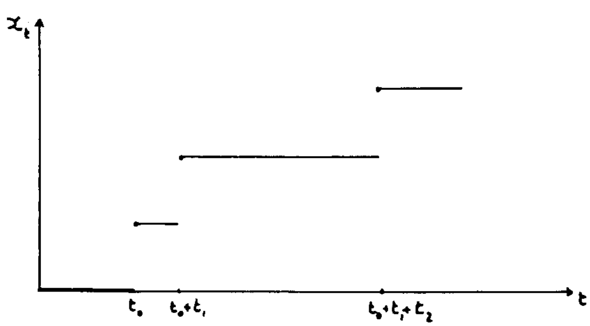

Fig. 1. A sample path and its holding times.

Holding times provide a convenient way of studying the asymptotic properties of the process:

Lemma 1. Suppose that (16) holds, but $w_j^{-1} < \infty$, $j = 0, 1, 2, \ldots$. Then for any $\alpha > 0$ the following relations hold almost surely:

$$\limsup_{t \to \infty} X_t/t^\alpha = \left(\liminf_{n \to \infty} n^{-1/\alpha} \sum_{j=0}^{n-1} T_j \right)^{-\alpha} \tag{17}$$

$$\liminf_{t \to \infty} X_t/t^\alpha = \left(\limsup_{n \to \infty} n^{-1/\alpha} \sum_{j=0}^{n-1} T_j \right)^{-\alpha} \tag{18}$$

In particular,

$$\lim_{t \to \infty} X_t/t^\alpha = \left(\lim_{n \to \infty} n^{-1/\alpha} \sum_{j=0}^{n-1} T_j \right)^{-\alpha} \tag{19}$$

whenever the limit in the right-hand side exists.

Proof. Under the conditions of the lemma,

$$T_j < \infty, \qquad \sum_{j=0}^{n-1} T_j \nearrow \infty, \qquad \mathbf{P}_{\underline{w}}\text{-almost surely} \tag{20}$$

so that

$$\limsup_{t \to \infty} X_t/t^\alpha = \limsup_{n \to \infty} X_{T_0 + \cdots + T_{n-1}}/(T_0 + \cdots + T_{n-1})^\alpha \tag{21}$$

$$\liminf_{t \to \infty} X_t/t^\alpha = \liminf_{n \to \infty} X_{T_0 + \cdots + T_{n-1}}/(T_0 + \cdots + T_{n-1})^\alpha \tag{22}$$

But

$$X_{T_0 + \cdots + T_{n-1}} = n \tag{23}$$

so that the conclusion follows by rewriting

$$n/(T_0 + \cdots + T_{n-1})^\alpha = (n^{-1/\alpha}(T_0 + \cdots + T_{n-1}))^{-\alpha} \quad \blacksquare \tag{24}$$

We note also for future reference the following properties of the holding times:

Proposition 1. Suppose that (i)

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} w_j^{-1} = a < \infty \tag{25}$$

and (ii)

$$\lim_{n \to \infty} n^{-2} \sum_{j=0}^{n-1} w_j^{-2} = 0 \tag{26}$$

Then

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} T_j = a \qquad \text{in probability} \tag{27}$$

that is to say

$$\forall \varepsilon > 0, \qquad \lim_{n \to \infty} \mathbf{P}_{\underline{w}} \left[\left| n^{-1} \sum_{j=0}^{n-1} T_j - a \right| < \varepsilon \right] = 1 \tag{28}$$

Proof. It is well known that *when the limiting random variable is degenerate*, convergence in probability is equivalent to convergence in distribution (see ref. 17, p. 260); hence it suffices to prove

$$\lim_{n \to \infty} \mathbf{E}_{\underline{w}} \left[\exp\left(-\lambda n^{-1} \sum_{j=0}^{n-1} T_j \right) \right] = \exp(-\lambda a) \tag{29}$$

for all $\lambda \geqslant 0$. But since the random variables T_j are independent,

$$\mathbf{E}_{\underline{w}} \left[\exp\left(-\lambda n^{-1} \sum_{j=0}^{n-1} T_j \right) \right] = \prod_{j=0}^{n-1} (1 + \lambda n^{-1} w_j^{-1})^{-1} \tag{30}$$

$$= \exp\left[-\sum_{j=0}^{n-1} \log(1 + \lambda n^{-1} w_j^{-1}) \right] \tag{31}$$

It is thus sufficient to prove

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} \log(1 + \lambda n^{-1} w_j^{-1}) = \lambda a \tag{32}$$

This in turn follows from the assumptions (25), (26) because

$$\begin{aligned} &\left| \lambda a - \sum_{j=0}^{n-1} \log(1 + \lambda n^{-1} w_j^{-1}) \right| \\ &\qquad \leqslant \lambda \left| a - n^{-1} \sum_{j=0}^{n-1} w_j^{-1} \right| \\ &\qquad\quad + \sum_{j=0}^{n-1} |\lambda n^{-1} w_j^{-1} - \log(1 + \lambda n^{-1} w_j^{-1})| &(33) \\ &\qquad \leqslant \lambda \left| a - n^{-1} \sum_{j=0}^{n-1} w_j^{-1} \right| + (\lambda^2 / 2n^2) \sum_{j=0}^{n-1} w_j^{-2} &(34) \end{aligned}$$

where we made use of the elementary inequality

$$0 \leqslant x - \log(1+x) \leqslant x^2/2, \qquad x \geqslant 0 \quad \blacksquare \tag{35}$$

It turns out that when the number a in (25) is 0 or infinity, condition (26) can be dispensed with:

Proposition 2. Suppose that for some $\gamma > 0$

$$\lim_{n \to \infty} n^{-\gamma} \sum_{j=0}^{n-1} w_j^{-1} = a \tag{36}$$

where a is either 0 or ∞. Then

$$\lim_{n \to \infty} n^{-\gamma} \sum_{j=0}^{n-1} T_j = a \qquad \text{in probability} \tag{37}$$

namely

$$\forall \epsilon > 0, \qquad \lim_{n \to \infty} \mathbf{P}_{\underline{w}} \left[n^{-\gamma} \sum_{j=0}^{n-1} T_j < \varepsilon \right] = 1 \qquad \text{if} \quad a = 0 \tag{38}$$

$$\forall \varepsilon > 0, \qquad \lim_{n \to \infty} \mathbf{P}_{\underline{w}} \left[n^{-\gamma} \sum_{j=0}^{n-1} T_j > \varepsilon \right] = 1 \qquad \text{if} \quad a = \infty \tag{39}$$

Proof. As in Proposition 1, we investigate

$$\mathbf{E}_{\underline{w}} \left[\exp\left(-\lambda n^{-\gamma} \sum_{j=0}^{n-1} T_j \right) \right] = \prod_{j=0}^{n-1} (1 + \lambda n^{-\gamma} w_j^{-1})^{-1} \tag{40}$$

But note that for any nonnegative numbers $a_0, a_1, ..., a_{n-1}$

$$1 + \sum_{j=0}^{n-1} a_j \leqslant \prod_{j=0}^{n-1} (1 + a_j) \leqslant \exp\left(\sum_{j=0}^{n-1} a_j \right) \tag{41}$$

so that

$$\begin{aligned} \exp\left(-\lambda n^{-\gamma} \sum_{j=0}^{n-1} w_j^{-1} \right) &\leqslant \mathbf{E}_{\underline{w}} \left[\exp\left(-\lambda n^{-\gamma} \sum_{j=0}^{n-1} T_j \right) \right] \\ &\leqslant \left(1 + \lambda n^{-\gamma} \sum_{j=0}^{n-1} w_j^{-1} \right)^{-1} \end{aligned} \tag{42}$$

Hence for any $\lambda \geqslant 0$

$$\lim_{n \to \infty} \mathbf{E}_{\underline{w}} \left[\exp\left(-\lambda n^{-\gamma} \sum_{j=0}^{n-1} T_j \right) \right] = \begin{cases} 1 & \text{if} \quad a = 0 \\ 0 & \text{if} \quad a = \infty \end{cases} \tag{43}$$

which is equivalent to (38), (39). $\blacksquare$

3. THE ASYMPTOTIC VELOCITY

Consider now a pure birth process where the transition rates are random variables $\{W_j\}_{j=0}^{\infty}$. We will assume that the W_j are independent identically distributed nonnegative random variables with no atom at the origin [see, however, Remark (i) after Proposition 1]. A first consequence of this assumption is that the "non-blow-up" condition (16) holds for almost all samples $\underline{w} = \{w_j\}_{j=0}^{\infty}$ of $\underline{W} = \{W_j\}_{j=0}^{\infty}$. Indeed, there must exist $0 < c < \infty$ such that $\mathbf{P}[W_0 < c] > 0$, so that

$$\sum_{j=0}^{\infty} \mathbf{P}[W_j^{-1} > c^{-1}] = \sum_{j=0}^{\infty} \mathbf{P}[W_0 < c] = \infty \tag{44}$$

and thus by the Borel–Cantelli lemma[17,18]

$$\mathbf{P}[W_j^{-1} > c^{-1} \text{ infinitely often}] = 1 \tag{45}$$

implying

$$\mathbf{P}\left[\sum_{j=0}^{\infty} W_j^{-1} = \infty\right] = 1 \tag{46}$$

By Lemma 1, the computation of $\lim_{t\to\infty} X_t/t$ reduces to that of $(\lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} T_j)^{-1}$. This last expression is obviously of the right form for the application of some law of large numbers. Such a procedure will indeed yield the correct result, but note that the blind application of Kolmogorov's strong law of large numbers for non-identically distributed random variables (see ref. 17, p. 364) to compute $\lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} T_j$ for a given sample $\underline{w}$ of $\underline{W}$ will require the condition

$$\sum_{j=1}^{\infty} j^{-2} \mathbf{D}_{\underline{w}}[T_{j-1}] < \infty, \qquad \text{i.e.,} \qquad \sum_{j=1}^{\infty} j^{-2} w_{j-1}^{-2} < \infty \tag{47}$$

For this condition to hold for almost every sample $\underline{w}$ we will need, by the two series theorem (see ref. 17, p. 361),

$$\sum_{j=1}^{\infty} j^{-4} \mathbf{D}[W_{j-1}^{-2}] < \infty \tag{48}$$

which holds whenever

$$\mathbf{E}[W_0^{-4}] < \infty \tag{49}$$

It turns out that condition (49) can be considerably weakened by using Loève's version of the strong law of large numbers (see Theorem 7 in the

Appendix and ref. 18, p. 253). Note for the sake of comparison that in ref. 4, $\mathbf{E}[W_0^{-q}]$ is assumed to exist for *all* $q \geqslant 0$.

Theorem 1. Suppose that for some $\varepsilon > 0$,

$$\mathbf{E}[W_0^{-1-\varepsilon}] < \infty \tag{50}$$

Then for almost all samples $\underline{w}$, almost all sample paths of X_t obey

$$\lim_{t \to \infty} X_t/t = (\mathbf{E}[W_0^{-1}])^{-1} \tag{51}$$

Proof. Take any realization $\underline{w}$ of $\underline{W}$; by Lemma 1 we have to prove

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} T_j = \mathbf{E}[W_0^{-1}], \qquad \mathbf{P}_{\underline{w}}\text{-almost surely} \tag{52}$$

By Theorem 7(ii) we have

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} (T_j - w_j^{-1}) = 0, \qquad \mathbf{P}_{\underline{w}}\text{-almost surely} \tag{53}$$

provided that for some $\alpha \geqslant 1$

$$\sum_{n=1}^{\infty} n^{-\alpha} w_{n-1}^{-\alpha} < \infty \tag{54}$$

Note the obvious identity

$$\mathbf{E}_{\underline{w}}[T_j^{\alpha}] = \int_0^{\infty} x^{\alpha} w_j e^{-w_j x}\, dx = w_j^{-\alpha} \Gamma(\alpha + 1) \tag{55}$$

By Theorem 7(i), condition (54) will hold for almost every sample $\underline{w}$ of $\underline{W}$ if for some $\beta < 1$

$$\sum_{n=1}^{\infty} n^{-\alpha\beta} \mathbf{E}[W_0^{-\alpha\beta}] < \infty \tag{56}$$

For this condition to hold it suffices to choose α, β so that $\alpha\beta = 1 + \varepsilon$; for instance,

$$\beta = 1 - \varepsilon, \qquad \alpha = \frac{1 + \varepsilon}{1 - \varepsilon} \tag{57}$$

Hence, under the single assumption (50), we have (53). On the other hand, by the strong law of large numbers for independent identically distributed random variables (see Theorem 8), we have

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} w_j^{-1} = \mathbf{E}[W_0^{-1}] \tag{58}$$

for almost samples $\underline{w}$ whenever $\mathbf{E}[W_0^{-1}] < \infty$. ▌

Remarks.

(i) The assumption that the random variables W_j are identically distributed can be relaxed and replaced by

$$\sum_{j=0}^{\infty} \mathbf{P}[W_j < \varepsilon] = \infty \qquad \text{for some} \quad \varepsilon > 0 \tag{59}$$

and

$$\sum_{j=1}^{\infty} j^{-1-\delta} \mathbf{E}[W_{j-1}^{-1-\delta}] < \infty \qquad \text{for some} \quad \delta > 0 \tag{60}$$

These two conditions ensure that (16) and (54) still hold almost surely. Theorem 1 remains valid provided that $\mathbf{E}[W_0^{-1}]$ is replaced by

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} \mathbf{E}[W_j^{-1}] \tag{61}$$

(ii) Obviously, (50) implies $\mathbf{E}[W_0^{-1}] < \infty$ because, if F is the probability distribution function of W_0, then

$$\begin{aligned} \mathbf{E}[W_0^{-1}] &= \int_{[0,\infty)} x^{-1}\, dF(x) \\ &= \int_{[0,1]} x^{-1}\, dF(x) + \int_{(1,\infty)} x^{-1}\, dF(x) && (62) \\ &\leqslant \int_{[0,1]} x^{-1-\varepsilon}\, dF(x) + \int_{(1,\infty)} dF(x) \\ &\leqslant \mathbf{E}[W_0^{-1-\varepsilon}] + 1 && (63) \end{aligned}$$

(iii) Since the original motivation for turning the transition rates into random variables was to account for the introduction of impurities in the system, it is instructive to compare the asymptotic velocity of the model with disorder to that of the classical model with transition rates equal to the mean value of the W_j. Since the W_j are identically distributed, the comparison random walk is a pure birth process where all the transition rates are equal to

$$w = \mathbf{E}[W_j] \tag{64}$$

and this is of course the Poisson process $\{N_t, t \geqslant 0\}$ with rate w. Its asymptotic velocity is well known to be

$$\lim_{t \to \infty} N_t/t = w = \mathbf{E}[W_0] \tag{65}$$

Since the function x^{-1}, $x > 0$, is convex, we see by Jensen's inequality that

$$(\mathbf{E}[W_0^{-1}])^{-1} \leqslant \mathbf{E}[W_0] \tag{66}$$

so that the disorder tends to slow down the process, as expected; see ref. 11 for a similar discussion. Much more drastic forms of slowing down will be investigated in the next section.

The assumption (50) can be further weakened if one is prepared to settle for a weaker form of convergence (although still almost sure with respect to $\underline{w}$):

Theorem 2. Suppose that $\mathbf{E}[W_0^{-1}] < \infty$. Then for almost all samples $\underline{w}$ the following limit holds:

$$\lim_{t \to \infty} X_t/t = (\mathbf{E}[W_0^{-1}])^{-1} \qquad \text{in probability} \tag{67}$$

Proof. By Proposition 1, it suffices to prove that the conditions

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} w_j^{-1} = \mathbf{E}[W_0^{-1}] \tag{68}$$

and

$$\lim_{n \to \infty} n^{-2} \sum_{j=0}^{n-1} w_j^{-2} = 0 \tag{69}$$

hold for almost every sample of $\underline{w}$ to conclude that

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} T_j = \mathbf{E}[W_0^{-1}] \qquad \text{in probability} \tag{70}$$

The first condition follows from the strong law of large numbers for independent identically distributed random variables (see Theorem 8). The second one follows from the Marcinkiewicz–Zygmund law of large numbers [see Theorem 9 with $X_j = (W_{j+1})^{-2}$, $\gamma = 1/2$].

To see that (70) implies (67), note that

$$\mathbf{P}_{\underline{w}}[X_t/t \leqslant (\mathbf{E}[W_0^{-1}])^{-1} - \varepsilon] = \mathbf{P}_{\underline{w}}[X_t \leqslant ((\mathbf{E}[W_0^{-1}])^{-1} - \varepsilon)t] \tag{71}$$

$$= \mathbf{P}_{\underline{w}}\left[\sum_{j=0}^{n(t)} T_j > t\right] \tag{72}$$

where $n(t)$ is the integer part

$$n(t) = [((\mathbf{E}[W_0^{-1}])^{-1} - \varepsilon)t] \tag{73}$$

Hence

$$\mathbf{P}_{\underline{w}}[X_t/t \leqslant (\mathbf{E}[W_0^{-1}])^{-1} - \varepsilon] = \mathbf{P}_{\underline{w}}\left[(1 + n(t))^{-1} \sum_{j=0}^{n(t)} T_j > t/(1 + n(t))\right] \tag{74}$$

tends to zero as $t \to \infty$ in view of (70), (73).

One can prove in the same way that the probability for X_t/t to exceed $(\mathbf{E}[W_0^{-1}])^{-1} + \varepsilon$ tends to zero for all $\varepsilon > 0$. ▮

4. THE SLOW APPROACH TO INFINITY WHEN $E[W_0^{-1}] = \infty$

It is natural to ask what happens when $\mathbf{E}[W_0^{-1}] = \infty$ (see refs. 10 and 11 for a discussion of this problem in related models). Aslangul *et al.*[4] study this question by assuming that W_0 has a probability density of the form

$$\frac{d}{dx}\mathbf{P}[W_0 \leqslant x] = f(x) = x^{\mu - 1} g(x), \qquad 0 < \mu < 1 \tag{75}$$

where g is some "cutoff function." We will introduce later a more general version of condition (75), but first we obtain a number of results that are independent of such a restriction. We first prove that if $\mathbf{E}[W_0^{-1}] = \infty$, no path of X_t can go to infinity as fast as t. This is based on the following simple observation:

Lemma 2. Let x_j, $j = 0, 1,..., n-1$, be nonnegative numbers and $0 \leqslant \alpha \leqslant 1$. Then

$$n^{\alpha - 1} \sum_{j=0}^{n-1} x_j^\alpha \leqslant \left(\sum_{j=0}^{n-1} x_j\right)^\alpha \leqslant \sum_{j=0}^{n-1} x_j^\alpha \tag{76}$$

Proof. To obtain the right-hand inequality, note that

$$x_j \Big/ \sum_{k=0}^{n-1} x_k \leqslant 1 \tag{77}$$

so that, since $0 \leqslant \alpha \leqslant 1$,

$$x_j \Big/ \sum_{k=0}^{n-1} x_k \leqslant \left(x_j \Big/ \sum_{k=0}^{n-1} x_k \right)^{\alpha} \tag{78}$$

Sum (78) over j to obtain the result. The left-hand inequality follows from the particular form of Hölder's inequality

$$\left(\sum_{j=0}^{n-1} a_j^{\alpha} b_j^{\alpha} \right)^{1/\alpha} \leqslant \left(\sum_{j=0}^{n-1} a_j \right) \left(\sum_{j=0}^{n-1} b_j^{\alpha/(\alpha-1)} \right)^{(1-\alpha)/\alpha} \tag{79}$$

with $a_j = x_j$, $b_j = 1$. ∎

Theorem 3. Suppose that $\mathbf{E}[W_0^{-1}] = \infty$, but that $\mathbf{E}[W_0^{-\alpha}] < \infty$ for all $0 < \alpha < 1$. Then for almost all realizations $\underline{w}$ of $\underline{W}$, the following limit holds:

$$\lim_{t \to \infty} X_t/t = 0, \qquad \mathbf{P}_{\underline{w}}\text{-almost surely} \tag{80}$$

Proof. Using the left-hand inequality of Lemma 2,

$$n^{-1} \sum_{j=0}^{n-1} T_j \geqslant \left(n^{-1} \sum_{j=0}^{n-1} T_j^{\gamma} \right)^{1/\gamma} \tag{81}$$

so that

$$\liminf_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} T_j \geqslant \left(\liminf_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} T_j^{\gamma} \right)^{1/\gamma} \tag{82}$$

But if we choose $0 < \gamma < 1$, we can prove as in Theorem 1 that almost surely

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} T_j^{\gamma} = \Gamma(\gamma+1)\, \mathbf{E}[W_0^{-\gamma}] \tag{83}$$

Hence

$$\liminf_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} T_j \geqslant \{\Gamma(\gamma+1)\, \mathbf{E}[W_0^{-\gamma}]\}^{1/\gamma} \tag{84}$$

This holds for all $\gamma < 1$. Take the supremum over all $\gamma < 1$ the right-hand side of (84) tends to infinity, proving

$$\liminf_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} T_j = \infty, \qquad \mathbf{P}_{\underline{w}}\text{-almost surely} \tag{85}$$

which is equivalent to (80). ∎

Since the rate of approach to infinity is not linear, we would like to characterize it. Our next theorem shows that X_t goes at least as fast as t^γ when $\mathbf{E}[W_0^{-\gamma}] = \infty$. As in the previous section, results that hold for almost all paths require slightly stronger assumptions than results that hold in probability:

Theorem 4. The following results hold for almost all samples $\underline{w}$:

(i) If $\mathbf{E}[W_0^{-\alpha}] < \infty$ for some $\alpha < 1$, then

$$\lim_{t \to \infty} X_t/t^\alpha = \infty \qquad \text{in probability} \tag{86}$$

(ii) If $\mathbf{E}[W_0^{-\alpha-\varepsilon}] < \infty$ for some $\alpha + \varepsilon < 1$, $\varepsilon > 0$, then

$$\lim_{t \to \infty} X_t/t^\alpha = \infty, \qquad \mathbf{P}_{\underline{w}}\text{-almost surely} \tag{87}$$

Proof. By Lemma 1 and Proposition 2, (i) will hold if for almost all samples $\underline{w}$

$$\lim_{n \to \infty} n^{-1/\alpha} \sum_{j=0}^{n-1} w_j^{-1} = 0 \tag{88}$$

But this is precisely the result guaranteed by the Marcinkiewicz–Zygmund law of large numbers when $\mathbf{E}[W_0^{-\alpha}] < \infty$ (see Theorem 9).

To prove (ii), take $\alpha < \gamma < \alpha + \varepsilon$. Using the right-hand inequality in Lemma 2, we obtain

$$\limsup_{n \to \infty} n^{-1/\gamma} \sum_{j=0}^{n-1} T_j \leqslant \left(\limsup_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} T_j^\gamma \right)^{1/\gamma} \tag{89}$$

But since $\gamma < \alpha + \varepsilon$ and $\mathbf{E}[W_0^{-\alpha-\varepsilon}] < \infty$, we can prove as in Theorem 1 that

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} T_j^\gamma = \Gamma(\gamma+1)\, \mathbf{E}[W_0^{-\gamma}] < \infty, \qquad \mathbf{P}_{\underline{w}}\text{-almost surely} \tag{90}$$

Hence

$$\liminf_{t \to \infty} X_t/t^{\gamma} \geqslant \{\Gamma(\gamma+1)\,\mathbf{E}[W_0^{-\gamma}]\}^{-1} > 0 \tag{91}$$

so that

$$\liminf_{t \to \infty} X_t/t^{\alpha} = \liminf_{t \to \infty} (X_t/t^{\gamma})(t^{\gamma-\alpha}) = \infty \quad \blacksquare \tag{92}$$

It remains to place an upper bound (better than that of Theorem 3) on the rate of approach to infinity of X_t. This turns out to be more delicate than the lower bound of Theorem 4, in the sense that it depends on the details of the probability distribution of W_0. However, the following partial result is independent of such details:

Proposition 3. Suppose that for some $\alpha > 0$, $\mathbf{E}[W_0^{-\alpha}] = \infty$. Then for almost all samples $\underline{w}$

$$\liminf_{t \to \infty} X_t/t^{\alpha} = 0, \qquad \mathbf{P}_{\underline{w}}\text{-almost surely} \tag{93}$$

Proof. We prove first that for almost all $\underline{w}$

$$\limsup_{n \to \infty} n^{-1/\alpha} \sum_{j=0}^{n-1} w_j^{-1} = \infty \tag{94}$$

This is because for any number c

$$\mathbf{P}\left[n^{-1/\alpha} \sum_{j=0}^{n-1} W_j^{-1} > c\right] \geqslant \mathbf{P}[n^{-1/\alpha} W_{n-1}^{-1} > c] \tag{95}$$

and the sum

$$\sum_{n=1}^{\infty} \mathbf{P}[n^{-1/\alpha} W_{n-1}^{-1} > c] = \sum_{n=1}^{\infty} \mathbf{P}[c^{-\alpha} W_0^{-\alpha} > n] \tag{96}$$

diverges to ∞ when $\mathbf{E}[W_0^{-\alpha}] = \infty$. Hence by the Borel–Cantelli lemma,[18]

$$\mathbf{P}[n^{-1/\alpha} W_{n-1}^{-1} > c \text{ infinitely often}] = 1 \tag{97}$$

and so

$$\mathbf{P}\left[n^{-1/\alpha} \sum_{j=0}^{n-1} W_j^{-1} > c \text{ infinitely often}\right] = 1 \tag{98}$$

implying that with probability one

$$\limsup_{n\to\infty} n^{-1/\alpha} \sum_{j=0}^{n-1} W_j^{-1} \geqslant c \tag{99}$$

But this holds for any c, proving (94).

Let now

$$L = \limsup_{n\to\infty} n^{-1/\alpha} \sum_{j=0}^{n-1} T_j \tag{100}$$

Because the random variables T_j are independent, the random variable L is degenerate (and possibly infinite) (see ref. 17, p. 358).

We get from (100)

$$e^{-\lambda L} = \liminf_{n\to\infty} \exp\left(-\lambda n^{-1/\alpha} \sum_{j=0}^{n-1} T_j\right) \tag{101}$$

so that by Fatou's lemma and formula (30)

$$\exp(-\lambda L) \leqslant \liminf_{n\to\infty} \mathbf{E}\left[\exp\left(-\lambda n^{-1/\alpha} \sum_{j=0}^{n-1} T_j\right)\right] \tag{102}$$

$$\leqslant \liminf_{n\to\infty} \left(1 + \lambda n^{-1/\alpha} \sum_{j=0}^{n-1} w_j^{-1}\right)^{-1} = 0 \tag{103}$$

Hence $L = \infty$. The result follows by Lemma 1. ▌

A complete description of the asymptotic behavior of X_t/t^α when $\mathbf{E}[W_0^{-\alpha}] = \infty$ requires a somewhat detailed knowledge of the distribution of W_0. For the rest of this section we will assume that

$$\mathbf{P}[W_0 \leqslant x] \sim x^\mu L(x) \qquad \text{as} \quad x \to 0+ \tag{104}$$

where $0 < \mu < 1$ and $L(x)$ is a function which *varies slowly at the origin*, namely (see ref. 15, Vol. II, p. 276)

$$\lim_{x\to 0+} L(xy)/L(x) = 1 \qquad \text{for all } y \tag{105}$$

Condition (104) is much more general than (75). Elementary manipulations show that (104) is equivalent to

$$1 - \mathbf{P}[W_0^{-1} \leqslant y] \sim y^{-\mu} L(y^{-1}) \qquad \text{as} \quad y \to +\infty \tag{106}$$

which by a standard Tauberian theorem (see ref. 15, Vol. II, pp. 445, 447) is itself equivalent to

$$1 - \mathbf{E}[e^{-\lambda W_0^{-1}}] \sim \Gamma(\mu+1)\, \lambda^\mu L(\lambda) \qquad \text{as} \quad \lambda \to 0+ \tag{107}$$

Let us first investigate the behavior of $n^{-1/\alpha}\sum_{j=0}^{n-1} W_j^{-1}$ under condition (104).

Proposition 4. Suppose that condition (104) holds, and let

$$b = \lim_{x \to 0+} L(x) \tag{108}$$

Then (i) if $\alpha > \mu$,

$$\lim_{n \to \infty} n^{-1/\alpha} \sum_{j=0}^{n-1} W_j^{-1} = \infty \qquad \text{almost surely} \tag{109}$$

(ii) if $0 \leqslant b < \infty$,

$$\limsup_{n \to \infty} n^{-1/\mu} \sum_{j=0}^{n-1} W_j^{-1} = \infty \qquad \text{almost surely} \tag{110}$$

$$\liminf_{n \to \infty} n^{-1/\mu} \sum_{j=0}^{n-1} W_j^{-1} = 0 \qquad \text{almost surely} \tag{111}$$

but

$$\lim_{n \to \infty} n^{-1/\mu} \sum_{j=0}^{n-1} W_j^{-1} = Y \qquad \text{in distribution} \tag{112}$$

where the random variable Y is characterized by

$$\mathbf{E}[e^{-\lambda Y}] = e^{-b\Gamma(\mu+1)\lambda^\mu} \tag{113}$$

and (iii) if $b = \infty$, (110) holds and

$$\lim_{n \to \infty} n^{-1/\mu} \sum_{j=0}^{n-1} W_j^{-1} = \infty \qquad \text{in distribution} \tag{114}$$

Proof. For $\lambda \geqslant 0$, $\alpha \geqslant \mu$, compute

$$\lim_{n \to \infty} \mathbf{E}\left[\exp\left(-\lambda n^{-1/\alpha} \sum_{j=0}^{n-1} W_j^{-1}\right)\right] = \lim_{n \to \infty} \{\mathbf{E}[\exp(-\lambda n^{-1/\alpha} W_0^{-1})]\}^n \tag{115}$$

Rewriting

$$\begin{aligned}
&\{\mathbf{E}[\exp(-\lambda n^{-1/\alpha} W_0^{-1})]\}^n \\
&\quad = (1 - \{1 - \mathbf{E}[\exp(-\lambda n^{-1/\alpha} W_0^{-1})]\})^n && (116)\\
&\quad = \left(1 - \frac{1 - \mathbf{E}[\exp(-\lambda n^{-1/\alpha} W_0^{-1})]}{\lambda^\mu n^{-\mu/\alpha} L(\lambda n^{-1/\alpha})\, \Gamma(\mu+1)} \lambda^\mu n^{-\mu/\alpha} L(\lambda n^{-1/\alpha})\, \Gamma(\mu+1)\right)^n && (117)
\end{aligned}$$

we see, using (107), that

$$\lim_{n\to\infty} \mathbf{E}\left[\exp\left(-\lambda n^{-1/\alpha}\sum_{j=0}^{n-1} W_j^{-1}\right)\right]$$
$$=\begin{cases} 0 & \text{if} \quad \alpha>\mu \\ \exp[-b\Gamma(\mu+1)\,\lambda^{\mu}] & \text{if} \quad \alpha=\mu, \quad 0\leqslant b<\infty \\ 0 & \text{if} \quad \alpha=\mu, \quad b=\infty \end{cases} \tag{118}$$

proving (112) and (114).

To conclude the proof, let

$$l=\liminf_{n\to\infty} n^{-1/\alpha}\sum_{j=0}^{n-1} W_j^{-1} \tag{119}$$

As the W_j^{-1} are independent random variables, l is degenerate. Moreover, for $\lambda\geqslant 0$

$$\limsup_{n\to\infty} \exp\left(-\lambda n^{-1/\alpha}\sum_{j=0}^{n-1} W_j^{-1}\right)=\exp(-\lambda l) \tag{120}$$

so that by Fatou's lemma

$$\exp(-\lambda l)\geqslant \limsup_{n\to\infty} \mathbf{E}\left[\exp\left(-\lambda n^{-1/\alpha}\sum_{j=0}^{n-1} W_j^{-1}\right)\right] \tag{121}$$

If $\alpha=\mu$ and $0\leqslant b<\infty$, this reads, by (118),

$$e^{-\lambda l}\geqslant e^{-b\Gamma(\mu+1)\,\lambda^{\mu}}, \qquad \lambda\geqslant 0 \tag{122}$$

hence

$$l\leqslant b\Gamma(\mu+1)\,\lambda^{\mu-1}, \qquad \lambda\geqslant 0 \tag{123}$$

This can hold for all λ's only if $l=0$. This proves (111); the proof of (110) is similar. Finally, in order to prove (109), note that (117) implies that when $\alpha>\mu$

$$\sum_{n=1}^{\infty} \{\mathbf{E}[\exp(-\lambda n^{-1/\alpha} W_0^{-1})]\}^n<\infty \tag{124}$$

Then *a fortiori* for any $c>0$

$$\sum_{n=1}^{\infty} \mathbf{P}\left[n^{-1/\alpha}\sum_{j=0}^{n-1} W_j^{-1}<c\right]<\infty \tag{125}$$

so that by the Borel–Cantelli lemma

$$\mathbf{P}\left[n^{-1/\alpha}\sum_{j=0}^{n-1} W_j^{-1} < c \text{ infinitely often}\right] = 0 \tag{126}$$

and thus

$$\liminf_{n\to\infty} n^{-1/\alpha}\sum_{j=0}^{n-1} W_j^{-1} > c, \qquad \text{almost surely} \tag{127}$$

Since this holds for every $c > 0$, (109) follows. ∎

Remark. When $b = \infty$, the behavior of

$$\liminf_{n\to\infty} n^{-1/\mu}\sum_{j=0}^{n-1} W_j^{-1}$$

depends on the details of the function L of formula (104); for instance, the finiteness or otherwise of

$$\sum_{n=1}^{\infty} \exp[-a^{-\mu}L(a^{-1}n^{-1/\mu})], \qquad a > 0$$

is relevant to this question.

It remains to use Proposition 4 to obtain results on X_t/t^α.

Theorem 5. Assume condition (104). Then the following results hold for almost every sample $\underline{w}$:

(i) If $\alpha > \mu$

$$\lim_{t\to\infty} X_t/t^\alpha = 0 \qquad \text{in probability} \tag{128}$$

(ii) If $0 \leqslant b \leqslant \infty$

$$\liminf_{t\to\infty} X_t/t^\mu = 0, \qquad \mathbf{P}_{\underline{w}}\text{-almost surely} \tag{129}$$

$$\limsup_{t\to\infty} X_t/t^\mu = \infty, \qquad \mathbf{P}_{\underline{w}}\text{-almost surely} \tag{130}$$

Proof. Property (128) follows Propositions 2 and 4(i). Property (129) follows from Proposition 3. In order to prove (130), let

$$l = \liminf_{n\to\infty} n^{-1/\mu}\sum_{j=0}^{n-1} T_j$$

We obtain, as in (120),

$$\exp(-\lambda l) = \limsup_{n\to\infty} \exp\left(-\lambda n^{-1/\mu} \sum_{j=0}^{n-1} T_j\right) \tag{131}$$

so that by Fatou's lemma, (42), and (111),

$$\exp(-\lambda l) \geqslant \limsup_{n\to\infty} \mathbf{E}_{\underline{w}}\left[\exp\left(-\lambda n^{-1/\mu} \sum_{j=0}^{n-1} T_j\right)\right] \tag{132}$$

$$\geqslant \limsup_{n\to\infty} \exp\left(-\lambda n^{-1/\mu} \sum_{j=0}^{n-1} w_j^{-1}\right) = 1 \tag{133}$$

This proves that $l = 0$, and the conclusion follows using Lemma 1. ∎

Remark. Theorem 5 contains no statement about the case $\alpha < \mu$; this is covered by Theorem 4 because under condition (104), $\mathbf{E}[W_0^{-\alpha}] < \infty$ when $\alpha < \mu$.

5. FLUCTUATIONS

Theorem 1 can be stated loosely as

$$X_t \sim t/\mathbf{E}[W_0^{-1}] \qquad \text{as} \quad t \to \infty \tag{134}$$

We investigate now corrections to that leading behavior. It would be natural to expect (by some central limit argument) that $t^{-1/2}(X_t - t/\mathbf{E}[W_0^{-1}])$ converges in distribution to a normal random variable. However, this is not so, the reason being that $t/\mathbf{E}[W_0^{-1}]$ is not a good enough approximation to $\mathbf{E}_{\underline{w}}[X_t]$ on the magnified scale that we are using now; see Remark (ii) after Theorem 6. It turns out that the function $p_{\underline{w}}(t)$ defined for any sample $\underline{w}$ of $\underline{W}$ by

$$p_{\underline{w}}(t) = \max\left\{n: \sum_{j=0}^{n-1} w_j^{-1} \leqslant t\right\} \tag{135}$$

is a sufficiently improved approximation. Before stating our central limit result, we note a few properties of $p_{\underline{w}}(t)$. Obviously, if (16) holds, $p_{\underline{w}}(t) \nearrow \infty$ as $t \nearrow \infty$. In fact, from (135)

$$\sum_{j=0}^{p_{\underline{w}}(t)-1} w_j^{-1} \leqslant t < \sum_{j=0}^{p_{\underline{w}}(t)} w_j^{-1} \tag{136}$$

so that for almost all realizations $\underline{w}$ of $\underline{W}$

$$\lim_{t\to\infty} t/p_{\underline{w}}(t) = \mathbf{E}[W_0^{-1}] \tag{137}$$

The fact that the upper bound of the sum in (136) is a random variable not independent of the summand causes no difficulty (see ref. 19, p. 13). Moreover, we have the following result.

Lemma 3. Suppose that $\mathbf{E}[W_0^{-2}] < \infty$; then for almost all realizations $\underline{w}$ of $\underline{W}$

$$\lim_{t\to\infty} t^{-1/2} \left| t - \sum_{j=0}^{p_{\underline{w}}(t)-1} w_j^{-1} \right| = 0 \tag{138}$$

Proof. From (136)

$$0 \leqslant t - \sum_{j=0}^{p_{\underline{w}}(t)-1} w_j^{-1} < w_{p_{\underline{w}}(t)}^{-1} \tag{139}$$

Hence, in view of (137) it suffices to prove

$$\lim_{n\to\infty} n^{-1/2} w_n^{-1} = 0 \qquad \text{almost surely} \tag{140}$$

By the Borel–Cantelli lemma this will follow from

$$\sum_{n=0}^{\infty} \mathbf{P}[W_n^{-1} > \varepsilon n^{1/2}] < \infty, \qquad \forall \varepsilon > 0 \tag{141}$$

which in turn follows from the assumption of the lemma, since

$$\sum_{n=0}^{\infty} \mathbf{P}[W_n^{-1} > \varepsilon n^{1/2}] = \sum_{n=0}^{\infty} \mathbf{P}[\varepsilon^{-2} W_0^{-2} > n] \tag{142}$$

$$\leqslant \mathbf{P}[\varepsilon^{-2} W_0^{-2} > 0] + \int_0^\infty \mathbf{P}[\varepsilon^{-2} W_0^{-2} > x]\, dx \tag{143}$$

$$= 1 + \varepsilon^{-2} \mathbf{E}[W_0^{-2}] \quad \blacksquare \tag{144}$$

We can now state and prove our main central limit result:

Theorem 6. Suppose that $\mathbf{E}[W_0^{-6}] < \infty$. Then for almost all realizations $\underline{w}$ of $\underline{W}$ the following limit holds:

$$\lim_{t\to\infty} \mathbf{P}_{\underline{w}}[t^{-1/2}(X_t - p_{\underline{w}}(t)) \leqslant x] = \Phi_{\sigma^2}(x) \tag{145}$$

where Φ_{σ^2} is the normal probability distribution functon defined in (9), (10).

Proof. Choose a realization $\underline{w}$ of $\underline{W}$; using (14), we see that

$$\mathbf{P}_{\underline{w}}[t^{-1/2}(X_t - p_{\underline{w}}(t)) \leqslant x] = \mathbf{P}_{\underline{w}}[X_t \leqslant xt^{1/2} + p_{\underline{w}}(t)] \tag{146}$$

$$= \mathbf{P}_{\underline{w}}\left[\sum_{j=0}^{N(t)-1} T_j > t\right] \tag{147}$$

where $N(t)$ stands for the following integer part:

$$N(t) = [xt^{1/2} + p_{\underline{w}}(t) + 1] \tag{148}$$

In preparation for the central limit theorem, we rewrite (146) in the form

$$\mathbf{P}_{\underline{w}}\left[\sum_{j=0}^{N(t)-1} (T_j - w_j^{-1})/\sigma_{N(t)} > \tau(t)\right] \tag{149}$$

where σ_n stands for the standard deviation

$$\sigma_n = \left(\sum_{j=0}^{n-1} w_j^{-2}\right)^{1/2} \tag{150}$$

and the argument $\tau(t)$ is

$$\tau(t) = \left(t - \sum_{j=0}^{N(t)-1} w_j^{-1}\right)\Big/\sigma_{N(t)} \tag{151}$$

Recall that for fixed $\underline{w}$ the random variables T_j are independent and exponentially distributed with parameter w_j. Hence the normal convergence of

$$\sum_{j=0}^{n-1} (T_j - w_j^{-1})/\sigma_n \tag{152}$$

as n tends to infinity is easily verified, either by checking that the Lindeberg condition holds[15] or by direct calculation:

$$\exp(-\lambda^2/2)\, \mathbf{E}_{\underline{w}}\left[\exp\left\{-\lambda \sum_{j=0}^{n-1} (T_j - w_j^{-1})/\sigma_n\right\}\right]$$

$$= \exp(-\lambda^2/2) \exp \sum_{j=0}^{n-1} [\lambda w_j^{-1}/\sigma_n - \log(1 + \lambda w_j^{-1}/\sigma_n)] \tag{153}$$

$$= \exp \sum_{j=0}^{n-1} [\lambda w_j^{-1}/\sigma_n - \lambda^2 w_j^{-2}/2\sigma_n^2 - \log(1 + \lambda w_j^{-1}/\sigma_n)] \tag{154}$$

The elementary inequality

$$-x^3/3 \leqslant x - x^2/2 - \log(1+x) \leqslant 0, \qquad x \geqslant 0 \tag{155}$$

implies for $\lambda \geqslant 0$

$$\exp\left(-\lambda^3 \sum_{j=0}^{n-1} w_j^{-3}/\sigma_n^3\right) \leqslant \exp(-\lambda^2/2)\, \mathbf{E}_{\underline{w}}\left[\exp\left\{-\lambda \sum_{j=0}^{n-1} (T_j - w_j^{-1})/\sigma_n\right\}\right] \leqslant 1 \tag{156}$$

Moreover,

$$\sum_{j=0}^{n-1} \frac{w_j^{-3}}{\sigma_n^3} = \frac{n^{-3/2} \sum_{j=0}^{n-1} w_j^{-3}}{(n^{-1} \sum_{j=0}^{n-1} w_j^{-2})^{3/2}} \tag{157}$$

converges to zero for almost any sample $\underline{w}$ by the Marcinkiewicz–Zygmund law of large numbers (see Theorem 9). Hence

$$\lim_{n \to \infty} \mathbf{E}_{\underline{w}}\left[\exp\left\{-\lambda \sum_{j=0}^{n-1} (T_j - w_j^{-1})/\sigma_n\right\}\right] = \exp(\lambda^2/2) \tag{158}$$

or equivalently

$$F_n(x) = \mathbf{P}\left[\sum_{j=0}^{n-1} (T_j - w_j^{-1})/\sigma_n \leqslant x\right] \to \Phi_1(x) \qquad \text{as} \quad n \to \infty \tag{159}$$

By (147), (149) we have, with F_n as in (159),

$$\mathbf{P}_{\underline{w}}[t^{-1/2}(X_t - p_{\underline{w}}(t)) \leqslant x] = 1 - F_{N(t)}(\tau(t)) \tag{160}$$

Obviously $N(t) \nearrow \infty$ as $t \nearrow \infty$. We analyze now the argument $\tau(t)$ [see (151)]:

$$\tau(t) = \frac{N^{-1/2}(t)[t - \sum_{j=0}^{N(t)-1} w_j^{-1}]}{[N^{-1}(t) \sum_{j=0}^{N(t)-1} w_j^{-2}]^{1/2}} \tag{161}$$

The denominator of (161) tends to $(\mathbf{E}[W_0^{-2}])^{1/2}$ for almost all $\underline{w}$ by the strong law of large numbers (Theorem 8). As for the numerator, rewrite it as follows:

$$t^{-1/2}\left(t - \sum_{j=0}^{p_{\underline{w}}(t)-1} w_j^{-1}\right) [t/N(t)]^{1/2} - N^{-1/2}(t) \sum_{j=p_{\underline{w}}(t)}^{N(t)-1} (w_j^{-1} - \mathbf{E}[W_0^{-1}]) + [N(t) - p_{\underline{w}}(t)]\, N^{-1/2}(t)\, \mathbf{E}[W_0^{-1}] \tag{162}$$

where we have assumed $x>0$, so that $N(t)>p_{\underline{w}}(t)$. The first term in (162) tends to zero by Lemma 3 and (148), (137). The second term converges to zero by Lemma 4 (see below) because there are only

$$N(t)-p_{\underline{w}}(t)\leqslant xt^{1/2}+1 \tag{163}$$

terms in the sum and $N^{1/2}(t)=O(t^{1/2})$. Finally, the last term is

$$-x[t/N(t)]^{1/2}\,\mathbf{E}[W_0^{-1}] \tag{164}$$

and converges by (148), (137) to

$$-x(\mathbf{E}[W_0^{-1}])^{3/2} \tag{165}$$

Thus, for almost all samples $\underline{w}$ we have

$$\lim_{t\to\infty}\tau(t)=-x(\mathbf{E}[W_0^{-1}])^{3/2}\,(\mathbf{E}[W_0^{-2}])^{-1/2} \tag{166}$$

The case $x<0$ is dealt with in a similar way. Moreover, the convergence of F_n to Φ_1 in (159) is known to be uniform (see ref. 17, p. 342), so that we can conclude from (160), (159), and (166) that for almost all samples $\underline{w}$

$$\lim_{t\to\infty}\mathbf{P}_{\underline{w}}[t^{-1/2}(X_t-p_{\underline{w}}(t))\leqslant x]=1-\Phi_1(-x(\mathbf{E}[W_0^{-1}])^{3/2}\,(\mathbf{E}[W_0^{-2}])^{-1/2})$$

$$=\Phi_{\sigma^2}(x) \tag{167}$$

with σ^2 as in (10). ▮

Remarks:

(i) The restriction $\mathbf{E}[W_0^{-6}]<\infty$ comes entirely from Lemma 4, which is presumably not optimal.

(ii) If instead of (145) we had attempted to compute, as seems natural,

$$\lim_{t\to\infty}\mathbf{P}_{\underline{w}}[t^{-1/2}(X_t-t/\mathbf{E}[W_0^{-1}])\leqslant x] \tag{168}$$

we would have been led to calculations similar to those of Theorem 6 with $N(t)$ and $\tau(t)$ replaced respectively by

$$M(t)=[xt^{1/2}+t/\mathbf{E}[W_0^{-1}]+1] \tag{169}$$

and

$$v(t)=\left\{tM(t)^{-1/2}-M^{-1/2}(t)\sum_{j=0}^{M(t)-1}w_j^{-1}\right\}\Big/M^{-1/2}(t)\,\sigma_{M(t)} \tag{170}$$

Obviously $M(t) \nearrow \infty$, so that, as in Theorem 6,

$$\lim_{t \to \infty} F_{M(t)}(x) = \Phi_1(x) \tag{171}$$

$$\lim_{t \to \infty} M^{-1/2}(t)\, \sigma_{M(t)} = (\mathbf{E}[W_0^{-2}])^{1/2} \tag{172}$$

But the numerator of (170) is

$$\{t - M(t)\,\mathbf{E}[W_0^{-1}]\}\, M^{-1/2}(t) - M^{-1/2}(t) \sum_{j=0}^{M(t)-1} (w_j^{-1} - \mathbf{E}[W_0^{-1}]) \tag{173}$$

The first term of (173) converges to $-x(\mathbf{E}[W_0^{-1}])^{3/2}$, but the second one must, by the law of the iterated logarithm (see ref. 17, p. 372), display huge fluctuations; namely, for almost every sample $\underline{w}$

$$\limsup_{t \to \infty} M^{-1/2}(t) \sum_{j=0}^{M(t)-1} (w_j^{-1} - \mathbf{E}[W_0^{-1}]) = \infty \tag{174}$$

$$\liminf_{t \to \infty} M^{-1/2}(t) \sum_{j=0}^{M(t)-1} (w_j^{-1} - \mathbf{E}[W_0^{-1}]) = -\infty \tag{175}$$

Consequently for almost all samples $\underline{w}$, and for all x

$$\limsup_{t \to \infty} \mathbf{P}_{\underline{w}}[t^{-1/2}(X_t - t/\mathbf{E}[W_0^{-1}]) \leqslant x] = 1 \tag{176}$$

$$\liminf_{t \to \infty} \mathbf{P}_{\underline{w}}[t^{-1/2}(X_t - t/\mathbf{E}[W_0^{-1}]) \leqslant x] = 0 \tag{177}$$

Lemma 4. Let Y_j, $j = 1, 2, 3,\ldots$, be independent identically distributed random variables with $\mathbf{E}[Y_j] = 0$, $\mathbf{E}[Y_j^6] < \infty$. Then

$$\lim_{n \to \infty} n^{-1/2} \sum_{j=n}^{n+1+\sqrt{n}} Y_j = 0 \qquad \text{almost surely} \tag{178}$$

Proof. It suffices to prove

$$\forall \varepsilon > 0, \qquad \mathbf{P}\left[\left| n^{-1/2} \sum_{j=n}^{n+1+\sqrt{n}} Y_j \right| > \varepsilon \text{ infinitely often}\right] = 0 \tag{179}$$

This will, by the Borel–Cantelli lemma, follow from

$$\sum_{n=1}^{\infty} \mathbf{P}\left[\left| n^{-1/2} \sum_{j=n}^{n+1+\sqrt{n}} Y_j \right| > \varepsilon\right] < \infty \tag{180}$$

But

$$\mathbf{P}\left[\left|n^{-1/2}\sum_{j=n}^{n+1+\sqrt{n}} Y_j\right|>\varepsilon\right]\leqslant\mathbf{E}\left[\left|\frac{n^{-1/2}}{\varepsilon}\sum_{j=n}^{n+1+\sqrt{n}} Y_j\right|^6\right] \tag{181}$$

$$=n^{-3}\varepsilon^{-6}\mathbf{E}\left[\left(\sum_{j=n}^{n+1+\sqrt{n}} Y_j\right)^6\right] \tag{182}$$

Expanding the argument and using $\mathbf{E}[Y_j]=0$, we get for the right-hand side of (182)

$$\begin{aligned} n^{-3}\varepsilon^{-6}\Bigg\{&\sum_{j=n}^{n+1+\sqrt{n}} \mathbf{E}[Y_1^6]+\frac{6!}{4!\,2!}\sum_{\substack{j,k=n\\ j\neq k}}^{n+1+\sqrt{n}} \mathbf{E}[Y_j^4]\,\mathbf{E}[Y_k^2] \\ &+\frac{6!}{3!\,3!}\sum_{\substack{j,k=n\\ j<k}}^{n+1+\sqrt{n}} \mathbf{E}[Y_j^3]\,\mathbf{E}[Y_k^3] \\ &+\frac{6!}{2!\,2!\,2!}\sum_{\substack{j,k,l=n\\ j<k<l}}^{n+1+\sqrt{n}} \mathbf{E}[Y_j^2]\,\mathbf{E}[Y_k^2]\,\mathbf{E}[Y_l^2]\Bigg\} \end{aligned} \tag{183}$$

$$\begin{aligned} =n^{-3}\varepsilon^{-6}\{&\mathbf{E}[Y_1^6]\,n^{-1/2}+15\mathbf{E}[Y_1^4]\,\mathbf{E}[Y_1^2]\,n^{1/2}(n^{1/2}-1) \\ &+20(\mathbf{E}[Y_1^3])^2\,n^{1/2}(n^{1/2}-1)/2 \\ &+120(\mathbf{E}[Y_1^2])^3\,n^{1/2}(n^{1/2}-1)(n^{1/2}-2)/6\} \end{aligned} \tag{184}$$

which is obviously summable since it is of order $n^{-3/2}$. ∎

APPENDIX. SOME STRONG LAWS OF LARGE NUMBERS

The following result can be extracted from ref. 18, p. 253:

Theorem 7 (Loève). Let X_n, $n=1, 2, 3,\ldots$, be a sequence of independent random variables; suppose that there exist numerical sequences $b_n \nearrow \infty$ and $0<r_n\leqslant 2$ such that

$$\sum_{n=1}^{\infty} b_n^{-r_n}\mathbf{E}[|X_n|^{r_n}]<\infty \tag{A1}$$

Then (i) $\sum_{n=1}^{\infty} b_n^{-1}(X_n-a_n)$ converges almost surely.

(ii) $b_n^{-1}\sum_{j=1}^{n}(X_j-a_j)$ converges almost surely to zero, where the numbers a_n are defined as follows:

$$a_n=\begin{cases}0 & \text{if } \ 0<r_n<1\\ \mathbf{E}[X_n] & \text{if } \ 1\leqslant r_n\leqslant 2\end{cases} \tag{A2}$$

If the random variables X_n are identically distributed, the standard convergence result is as follows (see ref. 17, p. 366):

Theorem 8 (Kolmogorov). Let X_n, $n = 1, 2, 3, \ldots$, be a sequence of independent identically distributed random variables with $\mathbf{E}[|X_n|] < \infty$. Then

$$\lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} X_j = \mathbf{E}[X_1] \qquad \text{almost surely} \tag{A3}$$

The following result sharpens Theorem 7 in the identically-distributed case (see ref. 20, p. 125):

Theorem 9 (Marcinkiewicz–Zygmund). Let X_n, $n = 1, 2, 3, \ldots$, be a sequence of independent identically distributed random variables. Suppose that for some $0 < \gamma < 2$, $\mathbf{E}[|X_n|^\gamma] < \infty$. Then

$$\lim_{n \to \infty} n^{-1/\gamma} \sum_{j=1}^{n} (X_j - a) = 0 \qquad \text{almost surely} \tag{A4}$$

where

$$a = \begin{cases} 0 & \text{if } 0 < \gamma < 1 \\ \mathbf{E}[X_1] & \text{if } 1 \leqslant \gamma < 2 \end{cases} \tag{A5}$$

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